Direct adaptive output-feedback control for unstable linear multidimensional systems with distinct input delays

Cong Vinh Tu, Natalia Dudarenko

***Abstract*— Existing algorithms for the direct compensation of unknown disturbances in unstable multivariable linear systems with input delays typically propose a single adaptive controller for the entire system; therefore, only one adaptive gain is used to adjust the overall convergence rate of all control channels. Moreover, these algorithms do not consider systems with output measurements only, where the state vector cannot be measured directly. This paper proposes the design of a new direct adaptive output-feedback control method for a class of unstable linear time-invariant multi-input multi-output systems with multiple distinct input delays, based on a combination of a state-feedback system decoupling approach and a direct disturbance compensation method using the internal model principle. The proposed method does not require time-domain integral calculations in the construction of the control law, thereby enhancing the robustness of the system. In addition, the adaptive algorithm can be designed independently for each control channel. As a result, the adaptive algorithm becomes easier to tune, negative effects caused by inter-channel interactions are eliminated, and the adaptive gains of each channel can be selected independently according to the desired performance requirements.**

***Keywords*— Adaptive control, Decoupling, Disturbance compensation, Falb–Wolovich approach, Control delays, Luenberger observer, Multidimensional system.**

I. Introduction

In practice, most industrial processes are multivariable systems with multiple control channels that exhibit complex internal interactions. These interactions, together with the presence of varying time delays between control channels, which are caused by factors such as mechanical delays in actuators or delays in the transmission of sensor data, present significant challenges in the design of control systems. In this context, the decoupling technique has been developed as an effective solution to eliminate unwanted cross interaction effects in multi-input, multi-output (MIMO) systems. The objective of decoupling is to ensure that each system output, after processing, depends only on its corresponding individual input. Decoupling approaches are commonly divided into two main types [1]: static methods and dynamic methods. Static decouplers usually employ gain matrices and are straightforward to implement, but their effectiveness is typically limited to steady state operation. In contrast, dynamic decouplers such as ideal decouplers, simplified decouplers, and inverse based decouplers are designed to handle both steady state and transient behaviors. However, these methods are often more complex, sensitive to inaccuracies in the system model, and face challenges when dealing with transmission zeros located in the right half of the complex plane. To address these difficulties, several dynamic decoupling techniques based on state feedback [2]–[5] have been proposed. Among them, the method developed using the Falb Wolovich approach [2] has been shown to be easy to implement and to deliver strong practical performance in decoupling, even in the presence of model uncertainties and external disturbances [6].

The paper focuses on the problem of adaptive disturbance compensation for MIMO linear time-invariant (LTI) systems. In this study, the reference signal and external disturbances are both unknown and take a harmonic form; they can be regarded as unknown disturbances acting on the control system. The problem of multiharmonic disturbance compensation has attracted attention due to its relevance in various engineering applications such as dynamic positioning of ships under sea disturbances, vertical landing on an oscillating platform, spacecraft stabilization, hard disk control, vibration attenuation, motor control, and others. An effective approach to address this problem is based on the internal model principle [7], in which the disturbance is modelled as the output of a linear autonomous system. To achieve zero steady-state error, the state and parameters of this system must be incorporated into the control law.

For the problem of harmonic disturbance compensation based on the internal model principle, several studies [8]–[12] have proposed estimating the key parameters of disturbances, such as frequency, amplitude, phase, and bias, and subsequently designing a control law based on these estimated values. The main advantage of this approach lies in decoupling the identification process from the controller, making it compatible with a wide range of control and disturbance rejection methods. However, a notable drawback is the requirement to maintain persistent excitation conditions for the regressor. In addition, a strategy for direct disturbance compensation for linear systems has been investigated. This method focuses on overcoming the difficulty of ensuring persistent excitation conditions for the regressor by designing a dedicated observer that estimates external disturbances using either the

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state vector or the system output. This method has been successfully applied in both continuous and discrete domains [13]–[21].

In relation to the problem of direct disturbance compensation for systems, in the case where disturbances may be unmatched in MIMO systems and may simultaneously affect both the state and the output of the system, study [18] implemented direct adaptive methods using the backstepping technique, while [19] designed an adaptive controller based on the certainty equivalence principle.

Extending the results of [19] to systems with identical input delays, works [20], [21] addressed the problem of direct adaptive disturbance compensation for unstable systems with distinct input delays. However, these studies only considered systems with measurable states. Therefore, the problem of disturbance compensation for systems with output measurements only, where the states are unmeasurable and the input delays are distinct, remains an open question.

In this research, a novel method is proposed for direct compensation of unknown external disturbances in tracking control for a class of unstable MIMO LTI systems with unmeasured states and distinct control delays. The approach begins with the construction of a Luenberger observer to estimate the system states. Then, a linear decoupling controller using state feedback is applied, developed based on the classical Falb–Wolovich method, to eliminate unwanted cross-coupling effects between inputs and outputs. Finally, the control law and adaptive algorithm are designed based on the parameterization of disturbances and the reference signal for the decoupled system. In this work, an adaptive algorithm with memory regressor extension (MRE) [20] is applied to improve the convergence of the adjusted parameters.

Notation: $\|x\|$ is the Euclidean norm of a vector x; L_1 is the space of integrally bounded vector functions in time domain; $blkdiag\{x_i\}$ is the block-diagonal matrix with diagonal elements x_i; $diag\{x_i\}$ is the diagonal matrix with diagonal elements x_i; $Col\{x_i\}$ is a column vector with elements x_i; $x \in \text{PE}$ means that a vector function $x(t)$ satisfies the persistent excitation (PE) condition [22].

II. Problem Statement

Consider a possibly unstable MIMO LTI plant affected by unknown external disturbances in the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu_d + D\omega(t), \\ y(t) = Cx(t) + Q\omega(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the unmeasurable state variable with the initial condition $x(0) = x_0$, and the control input vector with delayed components is defined as $u_d = [u_1(t-\tau_1), u_2(t-\tau_2), ..., u_m(t-\tau_m)]^\intercal$, with each τ_i is a known constant delay, and $u_i(t) = 0$ for $t < \tau_i \, (i = 1, 2, ..., m)$. The output vector is $y(t) \in \mathbb{R}^m$, and $\omega(t) \in \mathbb{R}^k$, denotes the unknown external disturbances. The system involves the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{n\times k}$, which are assumed to be known. Note that the matrix A is not necessarily stable, and the matrix $Q \in \mathbb{R}^{m\times k}$ may be unknown and $m \le n$.

The objective is to design a control law that satisfies the following requirements

$$\lim_{t\to\infty} \| y(t) - g(t) \| = 0, \tag{2}$$

where $g(t) \in \mathbb{R}^m$ is the vector of reference signals.

This paper is based on the following assumptions:

Assumption 1. The triple (A, B, C) is controllable and observable.

Assumption 2. The reference signal $g(t)$ is measurable. Each component $g_i(t)$ $(i = 1, 2, ..., m)$ of $g(t)$ is modeled as the output of a linear autonomous system

$$\dot{\xi}_{gi}(t) = \Gamma_{gi}\,\xi_{gi}(t),\, g_i(t) = h_{gi}^\intercal\,\xi_{gi}(t), \tag{3}$$

where $\xi_{gi}(t) \in \mathbb{R}^{q_{gi}}$ is an unmeasurable state with an unknown initial value $\xi_{gi}(0)$. The matrix Γ_{gi} is constant and unknown, with simple eigenvalues located on the imaginary axis, while $h_{gi}^\intercal$ is an unknown constant vector. Each pair (Γ_{gi}, h_{gi}) is observable, and the upper bounds of the state dimensions q_{gi} are known.

Assumption 3. Each component $\omega_\alpha(t)$ $(\alpha = 1, 2, \ldots, k)$ of $\omega(t)$ is assumed to be generated by a linear autonomous system

$$\dot{\xi}_{\omega\alpha}(t) = \Gamma_{\omega\alpha}\,\xi_{\omega\alpha}(t),\, \omega_\alpha(t) = h_{\omega\alpha}^\intercal\,\xi_{\omega\alpha}(t). \tag{4}$$

where $\xi_{\omega\alpha}(t) \in \mathbb{R}^{q_{\omega\alpha}}$ is an unmeasurable state with an unknown initial value $\xi_{\omega\alpha}(0)$. The matrix $\Gamma_{\omega\alpha}$ is constant and unknown, with simple eigenvalues located on the imaginary axis, while $h_{\omega\alpha}^\intercal$ is an unknown constant vector. Each pair $(\Gamma_{\omega\alpha}, h_{\omega\alpha})$ is observable, and the upper bounds of the state dimensions $q_{\omega\alpha}$ are known.

III. Design of the Decoupler and the State Observer

To decouple system (1) using dynamic state feedback, we construct the control law [2] in the following form:

$$u_f(t) = -K\hat{x}(t) + Fu_d, \tag{5}$$

where K and F are real constant matrices of appropriate size and $\hat{x}(t)$ is the estimated state obtained from a Luenberger-type observer [23]

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_f(t) + L\big(y(t) - C\hat{x}(t)\big). \tag{6}$$

Here, $L \in \mathbb{R}^{n\times m}$ is the observer gain matrix.

Consider the matrix C described in the form $C = [c_1, c_2, \ldots, c_m]^\intercal$ and define the integers σ_i, for $i = 1, 2, ..., m$ referred to as the order difference of the output, so that

$$\sigma_i = \begin{cases} \min\left(l \mid c_i A^{l-1} B \neq 0^T\right), & l = 1, 2, \ldots, n-1, \\ n-1, & \text{if } c_i A^{l-1} B = 0^T, \; l = 1, 2, \ldots, n. \end{cases} \quad (7)$$

Here, $0^T \in \mathbb{R}^m$ is a row vector with elements equal to zero.

We represent the desired transfer functions of the subsystems obtained after the decoupling process of system (1) as follows:

$$G_{ii}(s) = \frac{q_{i0}}{q_{i0} + q_{i1}s + q_{i2}s^2 + \cdots + s^{\sigma_i}} = \frac{q_{i0}}{\prod_{j=1}^{\sigma_i}(s - \lambda_{ij})},$$

where s is the Laplace operator, λ_{ij} are the desired eigenvalues.

By defining

$$B^* = \begin{bmatrix} c_1 A^{\sigma_1 - 1} B \\ c_2 A^{\sigma_2 - 1} B \\ \vdots \\ c_m A^{\sigma_m - 1} B \end{bmatrix}, C^* = \begin{bmatrix} c_1 A^{\sigma_1} + \sum_{z=0}^{\sigma_1 - 1} q_{1z} c_1 A^z \\ \vdots \\ c_m A^{\sigma_m} + \sum_{z=0}^{\sigma_m - 1} q_{mz} c_m A^z \end{bmatrix},$$

the following proposition [6] can be stated.

Proposition 1. System (1) is decouplable if and only if the matrix B^* is non-singular, if this condition is satisfied, by choosing $F = (B^*)^{-1} diag\{q_{i0}\}$, $K = (B^*)^{-1} C^*$, for the decoupling control law (5), the result is a diagonal-form closed-loop transfer function matrix, represented as follows:

$$G(s) = diag\{G_{ii}(s)\}, \quad (8)$$

where $G(s) = C(sI - A_f)^{-1} B_f$ with $A_f = A - BK$ and $B_f = BF$.

Remark 1. The system (1), after being decoupled by the control law constructed using the matrices K and F as stated in Proposition 1, is equivalent to m independent single-input single-output systems and can be represented in the following state-space form

$$\begin{cases} \dot{x}(t) = A_f x(t) + B_f u_d + D\omega(t) + \upsilon, \\ y(t) = Cx(t) + Q\omega(t). \end{cases} \quad (9)$$

Here, the matrix A_f is Hurwitz, and υ decay exponentially.

IV. Reference signal parametrization

Exosystem (3) can be represented in the canonical form [19]

$$\dot{\bar{\xi}}_{gi}(t) = G_{gi}\bar{\xi}_{gi}(t) + l_{gi}\, g_i(t), g_i(t) = \theta_{gi}^T \bar{\xi}_{gi}(t), \quad (10)$$

where $\bar{\xi}_{gi}(t) \in \mathbb{R}^{q_{gi}}$ is the state vector with a certain initial conditions $\bar{\xi}_{gi}(0)$; $\theta_{gi} \in \mathbb{R}^{q_{gi}}$ is the vector of unknown parameters; $G_{gi} \in \mathbb{R}^{q_{gi} \times q_{gi}}$ is the Hurwitz matrix, l_{gi} is the constant vector so that the pair (G_{gi}, l_{gi}) is controllable.

The exosystem (10) can be represented in the block-diagonal form as

$$\dot{\bar{\xi}}_g(t) = G_g\, \bar{\xi}_g(t) + L_g\, g(t), g(t) = \theta_g^T\, \bar{\xi}_g(t), \quad (11)$$

or

$$\dot{\bar{\xi}}_g(t) = (G_g + L_g \theta_g^T)\bar{\xi}_g(t), g(t) = \theta_g^T\, \bar{\xi}_g(t), \quad (12)$$

where $\bar{\xi}_g(t) = [\bar{\xi}_{g1}^T(t), \bar{\xi}_{g2}^T(t), \ldots, \bar{\xi}_{gm}^T(t)]^T \in \mathbb{R}^{q_g}$; $\theta_g^T = blkdiag\{\theta_{gi}^T\} \in \mathbb{R}^{m \times q_g}$; $G_g = blkdiag\{G_{gi}\} \in \mathbb{R}^{q_g \times q_g}$; $L_g = blkdiag\{l_{gi}\} \in \mathbb{R}^{q_g \times m}$.

The fundamental solution of equation (12) can be used to predict the value of $\bar{\xi}_g(t)$, which can be represented as follows:

$$\bar{\xi}_g(t) = \Phi_{gi}\bar{\xi}_g(t - \tau_i), \quad (13)$$

where $\Phi_{gi} = e^{\left(G_g + L_g \theta_g^T\right)\tau_i} \in \mathbb{R}^{q_g \times q_g}$.

Since $g(t)$ is measurable, based on expression (11) we can construct the reference observer in the form

$$\dot{\hat{\xi}}_g(t) = G_g\, \hat{\xi}_g(t) + L_g\, g(t), g(t) = \theta_g^T\, \hat{\xi}_g(t) + \upsilon_r, \quad (14)$$

with arbitrary initial condition $\hat{\xi}_g(0)$, then $\bar{\xi}_g(t) = \hat{\xi}_g(t) + \upsilon_g$. Here, υ_r and υ_g decay exponentially.

V. Disturbance parametrization

Exosystem (4) can be represented in the canonical form

$$\dot{\bar{\xi}}_{\omega\alpha}(t) = G_{\omega\alpha}\bar{\xi}_{\omega\alpha}(t) + l_{\omega\alpha}\,\omega_\alpha(t), \omega_\alpha(t) = \theta_{\omega\alpha}^T \bar{\xi}_{\omega\alpha}(t), \quad (15)$$

where $\bar{\xi}_{\omega\alpha}(t) \in \mathbb{R}^{q_{\omega\alpha}}$ is the state vector with a certain initial conditions $\bar{\xi}_{\omega\alpha}(0)$; $\theta_{\omega\alpha} \in \mathbb{R}^{q_{\omega\alpha}}$ is the vector of unknown parameters; $G_{\omega\alpha} \in \mathbb{R}^{q_{\omega\alpha} \times q_{\omega\alpha}}$ is the Hurwitz matrix, $l_{\omega\alpha}$ is the constant vector so that the pair $(G_{\omega\alpha}, l_{\omega\alpha})$ is controllable.

The exosystem (15) can be represented in the block-diagonal form as

$$\dot{\bar{\xi}}_\omega(t) = G_\omega \bar{\xi}_\omega(t) + L_\omega \omega(t), \omega(t) = \theta_\omega^T\, \bar{\xi}_\omega(t), \quad (16)$$

or

$$\dot{\bar{\xi}}_\omega(t) = \left(G_\omega + L_\omega \theta_\omega^T\right)\bar{\xi}_\omega(t), \omega(t) = \theta_\omega^T\, \bar{\xi}_\omega(t), \quad (17)$$

where $\bar{\xi}_\omega(t) = [\bar{\xi}_{\omega 1}^T(t), \bar{\xi}_{\omega 2}^T(t), \ldots, \bar{\xi}_{\omega k}^T(t)]^T \in \mathbb{R}^{q_\omega}$; $\theta_\omega^T = blkdiag\{\theta_{\omega\alpha}^T\} \in \mathbb{R}^{k \times q_\omega}$; $G_\omega = blkdiag\{G_{\omega\alpha}\} \in \mathbb{R}^{q_\omega \times q_\omega}$; $L_\omega = blkdiag\{l_{\omega\alpha}\} \in \mathbb{R}^{q_\omega \times k}$.

The fundamental solution of equation (17) can be used to predict the value of $\bar{\xi}_\omega(t)$, which can be represented as follows:

$$\bar{\xi}_\omega(t) = \Phi_{\omega i}\bar{\xi}_\omega(t - \tau_i), \quad (18)$$

where $\Phi_{\omega i} = e^{\left(G_\omega + L_\omega \theta_\omega^T\right)\tau_i} \in \mathbb{R}^{q_\omega \times q_\omega}$.

Remark 1. From (13) and (18), the state $\bar{\xi}(t)$ of the exosystem can be expressed via its delayed value as follows:

$$\bar{\xi}(t) = \Phi_i \bar{\xi}(t - \tau_i), \quad (19)$$

where $\bar{\xi}(t - \tau_i) = [\bar{\xi}_g^T(t - \tau_i), \bar{\xi}_\omega^T(t - \tau_i)]^T$, and $\Phi_i = blkdiag\left\{\Phi_{gi}, \Phi_{\omega i}\right\}$.

Based on (16), we design a disturbance observer [20] used to estimate the state $\bar{\xi}_\omega$. Here, the system state $x(t)$ is replaced by its estimate $\hat{x}(t)$ obtained from the Luenberger observer (6). We construct the disturbance observer in the form:

$$\begin{cases}\dot{\varphi}(t) = G_\omega \varphi(t) + (G_\omega N - NA_f)\hat{x}(t) - NB_f u_d, \\ \hat{\xi}_\omega(t) = \varphi(t) + N\hat{x}(t),\end{cases} \tag{20}$$

where $\hat{\xi}_\omega(t)$ is the vector estimation of $\bar{\xi}_\omega(t)$; $\varphi(t) \in \mathbb{R}^{q_\omega}$ is the auxiliary observer vector; for the matrix $N = [N_1, N_2, \ldots, N_k]^{\bullet} \in \mathbb{R}^{q_\omega \times n}$, the equality holds:

$$N_\alpha D = L_{0\alpha},$$

where $N_\alpha \in \mathbb{R}^{q_{\omega\alpha} \times n}$ is the observer number corresponding to the external disturbance, and the matrix $L_{0\alpha}$:

$$L_{0\alpha} = [0_{q_{\omega\alpha}}, \ldots, 0_{q_{\omega\alpha}}, l_{\omega\alpha}, 0_{q_{\omega\alpha}}, \ldots, 0_{q_{\omega\alpha}}],$$

where the vector $l_{\omega\alpha}$ is as the α-th column; $0_{q_{\omega\alpha}}$ is the zero vector of dimension $q_{\omega\alpha}$.

Then $\bar{\xi}_\omega(t) = \hat{\xi}_\omega(t) + \upsilon_\omega$, while υ_ω exponentially decays.

VI. Error model and control law design

To transform the system model (9) into a form where the signals produced by the exosystems are matched with the control input u_d, we use the matrix regulator equation [19]. Under Assumption 1, 2 and 3, for system (9), there exist matrices $M_g \in \mathbb{R}^{n \times q_g}$ and $\psi_g \in \mathbb{R}^{q_g \times m}$ that satisfy the following matrix equations

$$B_f \psi_g^{\bullet} = A_f M_g - M_g(G_g + L_g \theta_g^{\bullet}), CM_g = \theta_g^{\bullet}. \tag{21}$$

Moreover, there exist matrices $M_\omega \in \mathbb{R}^{n \times q_\omega}$ and $\psi_\omega \in \mathbb{R}^{q_\omega \times m}$ that satisfy the following matrix equations

$$B_f \psi_\omega^{\bullet} = A_f M_\omega - M_\omega(G_\omega + L_\omega \theta_\omega^{\bullet}) + D\theta_\omega^{\bullet}, CM_\omega = -Q\theta_\omega^{\bullet}. \tag{22}$$

We define the state error as $e(t) = x(t) - M\bar{\xi}(t)$, and the output tracking error as $\delta(t) = y(t) - g(t)$, where $M = [M_g, M_\omega]$ and $\bar{\xi}(t) = [\bar{\xi}_g(t), \bar{\xi}_\omega(t)]^{\bullet}$. From the matrix equations (9), (12), (17), (21) and (22), we rewrite the system as

$$\begin{cases}\dot{e}(t) = A_f e(t) + B_f(\psi^{\bullet}\bar{\xi}(t) + u_d) + \upsilon, \\ \delta(t) = Ce(t).\end{cases} \tag{23}$$

Here, the matrix $\psi^{\bullet} = [\psi_g^{\bullet}, \psi_\omega^{\bullet}]$ is unknown.

Since the transfer function matrix $G(s)$ is diagonal, the control signal u and the output of the exosystem have the same delays. From equation (19), we can write the following:

$$\psi^{\bullet}\bar{\xi}(t) = \begin{bmatrix}\psi_1^{\bullet}\bar{\xi}(t) \\ \psi_2^{\bullet}\bar{\xi}(t) \\ \vdots \\ \psi_m^{\bullet}\bar{\xi}(t)\end{bmatrix} = \begin{bmatrix}\psi_1^{\bullet}\Phi_1\bar{\xi}(t-\tau_1) \\ \psi_2^{\bullet}\Phi_2\bar{\xi}(t-\tau_2) \\ \vdots \\ \psi_m^{\bullet}\Phi_m\bar{\xi}(t-\tau_m)\end{bmatrix} = \eta^{\bullet}\bar{\xi}_d,$$

where $\bar{\xi}_d = [\bar{\xi}^{\bullet}(t-\tau_1), \bar{\xi}^{\bullet}(t-\tau_2), \ldots, \bar{\xi}^{\bullet}(t-\tau_m)]^{\bullet}$, $\eta^{\bullet}$ is the new matrix of unknown parameters, while $\psi_i^{\bullet}$ is the i-th row of the matrix $\psi^{\bullet}$.

Based on the reference observer (14) and the disturbance observer (20), the unmeasured exosystem state $\bar{\xi}(t)$ is replaced by its estimate $\hat{\xi}(t)$ neglecting the exponentially decaying term since $G(s)$ is stable. Then, we have:

$$\begin{cases}\dot{e}(t) = A_f e(t) + B_f[\eta^{\bullet}\hat{\xi}_d + u_d], \\ \delta(t) = Ce(t).\end{cases} \tag{24}$$

From the matrix A_f is Hurwitz, we can define the following control law for system (24) with distinct input delays:

$$u(t) = -\hat{\eta}^{\bullet}\hat{\xi}(t), \tag{25}$$

where $\hat{\eta}^{\bullet} = [\hat{\eta}_1^{\bullet}, \hat{\eta}_2^{\bullet}, \ldots, \hat{\eta}_m^{\bullet}]^{\bullet} \in \mathbb{R}^{m \times q}$ is a coefficient matrix determined by the adaptive algorithm, which is proposed below. Substituting (25) into (24), we obtain

$$\begin{cases}\dot{e}(t) = A_f e(t) + B_f\tilde{\eta}_d^{\bullet}\hat{\xi}_d, \\ \delta(t) = Ce(t).\end{cases} \tag{26}$$

Here, $\hat{\eta}_d^{\bullet} = [\hat{\eta}_1^{\bullet}(t-\tau_1), \hat{\eta}_2^{\bullet}(t-\tau_2), \ldots, \hat{\eta}_m^{\bullet}(t-\tau_m)]^{\bullet}$ is the delayed matrices of the adjustable parameters, and $\tilde{\eta}_d^{\bullet} = \eta^{\bullet} - \hat{\eta}_d^{\bullet}$.

VII. Design of adaptation algorithm

From (26), it should be noted that the negative effect of the delay has not been completely eliminated. Let us construct the augmented state vector [13]

$$\hat{e} = e + \chi, \tag{27}$$

where the signal χ is defined

$$\dot{\chi} = A_f\chi + B_f(\hat{\eta}_d^{\bullet} - \hat{\eta}^{\bullet})\hat{\xi}_d. \tag{28}$$

Differentiating expression (27), taking into account (26) and (28), we obtain

$$\begin{cases}\dot{\hat{e}}(t) = A_f\hat{e}(t) + B_f\tilde{\eta}^{\bullet}\hat{\xi}_d, \\ \varepsilon(t) = C\hat{e}(t).\end{cases} \tag{29}$$

Let us rewrite (29) in the following form:

$$\varepsilon = G(s)[\tilde{\eta}^{\bullet}\hat{\xi}_d]. \tag{30}$$

For the purpose of applying the adaptation algorithm, from (30) the augmented error can be defined according to the swapping scheme [22] as follows:

$$\bar{\varepsilon} = \varepsilon + G(s)[\hat{\eta}^{\bullet}\hat{\xi}_d] - \Delta^{\bullet}\hat{\eta}. \tag{31}$$

where

$$\Delta^{\bullet} = \begin{bmatrix}G_{11}(s)[\hat{\xi}^{\bullet}(t-\tau_1)] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_{mm}(s)[\hat{\xi}^{\bullet}(t-\tau_m)]\end{bmatrix}.$$

Then the following equality is valid:

$$\bar{\varepsilon} = \Delta^{\bullet}\tilde{\eta}. \tag{32}$$

Proposition 2. Under Assumptions 1, 2, and 3, if system (1) satisfies the state-feedback decoupling condition from Proposition 1, then after performing the decoupling through the control law (5), the adaptive algorithm with MRE can be created independently for each control channel as follows:

$$\dot{\hat{\eta}}_i = \mu_i(Y_i - \Omega_{ii}\hat{\eta}_i), i = 1, 2, \ldots, m. \tag{33}$$

Here, $Y_i = H(s)[\Delta_i\hat{\varepsilon}_i]$, $\Omega_i = H(s)[\Delta_i\Delta_i^{\bullet}]$, $\Delta_i^{\bullet} = G_{ii}(s)[\hat{\xi}^{\bullet}(t-\tau_i)]$, $\hat{\varepsilon}_i = \delta_i - G_{ii}(s)[u_i(t-\tau_i)]$, where δ_i is a component of δ, and $H(s) = \frac{1}{\beta s + 1}$ with $\beta > 0$.

Proof. From expression (32), we introduce an extended variable

$$\hat{\varepsilon} = \bar{\varepsilon} + \Delta^{\cdot}\,\hat{\eta}. \qquad (34)$$

In this construction of the extended variable, we observe that the regressor matrix $\Delta^{\cdot}$ has a diagonal configuration. The diagonal components correspond to delayed regressors, which are filtered according to the dynamics of each control channel, as expressed in the following equation:

$$\Delta_i^{\cdot} = G_{ii}(s)[\hat{\xi}^{\cdot}(t-\tau_i)], i = 1, 2, \ldots, m.$$

By taking into account (24), (30), and (31), equation (34) is rewritten:

$$\hat{\varepsilon} = \delta - G(s)[u_d], \qquad (35)$$

or

$$\hat{\varepsilon} = \Delta^{\cdot}\,\eta. \qquad (36)$$

By multiplying both sides of equation (36) by Δ and applying the transfer function $H(s) = \frac{1}{\beta s + 1}$ with $\beta > 0$, we obtain:

$$H(s)[\Delta\hat{\varepsilon}] = H(s)[\Delta\Delta^{\cdot}\,]\eta, \qquad (37)$$

Based on equation (37), we can construct an adaptation algorithm with MRE [22] as follows:

$$\dot{\hat{\eta}} = \mu(Y - \Omega\hat{\eta}), \qquad (38)$$

where $Y = H(s)[\Delta\hat{\varepsilon}]$, $\Omega = H(s)[\Delta\Delta^{\cdot}\,]$, and $\mu > 0$ is the adaptation gain.

The vector Y and the matrix Ω which are constructed from the matrix Δ with a diagonal structure, can thus be written in the following form:

$$Y = Col\left\{H(s)[\Delta_i\hat{\varepsilon}_i]\right\}, i = 1, 2, \ldots, m. \qquad (39)$$

$$\Omega = blkdiag\left\{H(s)[\Delta_i\Delta_i^{\cdot}\,]\right\}, i = 1, 2, \ldots, m. \qquad (40)$$

In view of equations (38), (39), and (40), we obtain the adaptive algorithm for each control channel:

$$\dot{\hat{\eta}}_i = \mu_i(Y_i - \Omega_{ii}\hat{\eta}_i), \qquad (41)$$

where $Y_i = H(s)[\Delta_i\hat{\varepsilon}_i]$, $\Omega_i = H(s)[\Delta_i\Delta_i^{\cdot}\,]$, and $\hat{\varepsilon}_i = \delta_i - G_{ii}(s)[u_i(t-\tau_1)]$.

Based on expression (41), Proposition 2 has been proved.

The algorithm of adaptation (38) ensures the following closed-loop system properties.

Lemma 1. Under Assumptions 1, 2, and 3, if system (1) satisfies the state-feedback decoupling condition stated in Proposition 1, then the control law (25), together with the reference signal observer (14), the disturbance observer (20), the swapping scheme (31), and the adaptive algorithm (33), when applied to the plant (1), yields the following results:

L1.1. the boundedness of $\|\bar{\varepsilon}\|$, $\|\hat{\eta}\|$;

L1.2. the asymptotic convergence $\|\dot{\hat{\eta}}(t)\| \to 0, \|\Delta^{\cdot}\,(t)\tilde{\eta}(t)\| \to 0$ $\|\Delta^{\cdot}\,(t)\tilde{\eta}_d(t)\| \to 0$ as $t \to \infty$;

L1.3. if $\lambda_K(t) \notin \mathrm{L}_1$, where $\lambda_K(t)$ is the minimum eigenvalue of Ω, then $\|\tilde{\eta}\| \to 0$ as $t \to \infty$;

L1.4. if $\Delta \in \mathrm{PE}$, then $\|\tilde{\eta}(t)\|$ tends to zero exponentially, and this convergence can be accelerated by increasing μ.

Proof. See Chapter 3 in [22]

Thus, it can be concluded that the control objective (2) is achievable with the proposed method, and the following theorem can be stated.

Theorem 1. Under Assumptions 1, 2, and 3, if system (1) meets the state-feedback decoupling condition outlined in Proposition 1, then applying the state observer (6) to system (1), combined with the control law (25), the reference observer (14), the disturbance observer (20), the swapping scheme (31), and the adaptation algorithm (33) to system (9), which is derived from system (1) by applying the control law (5), guarantees that all signals in the closed-loop system are bounded, and the control objective (2) is achieved.

VIII. NUMERICAL EXAMPLE

Consider the system (1) with the first input delay $\tau_1 = 1s$, and the second input delay $\tau_1 = 3s$, the matrices

$$A = \begin{bmatrix} -3.2 & -1.7 & -3 \\ 2.8 & 1.3 & 4 \\ 0.3 & 0.3 & -0.5 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & -3 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

We assume that the reference signal and the external disturbance have a harmonic form

$$g(t) = \begin{bmatrix} 3\sin(2t) \\ 2 \end{bmatrix}, \omega(t) = \begin{bmatrix} 1\sin(1t) \\ 1 + 2\sin(2t) \end{bmatrix},$$

with unknown frequency, amplitude, phase, and bias.

In designing the Luenberger observer (6), the eigenvalues -2, -3 and -4 are selected. Using the pole placement technique, the observer gain matrix L is calculated as:

$$L = \begin{bmatrix} -8.0988 & 4.4574 \\ 10.6506 & -5.0619 \\ 2.7762 & -1.1783 \end{bmatrix}.$$

From (7), we obtain $\sigma_1 = 2$ and $\sigma_2 = 1$. By choosing the poles as $\lambda_{11} = -6, \lambda_{12} = -2$ and $\lambda_{21} = -4$. The decoupling control law is constructed using the matrices

$$F = \begin{bmatrix} 12 & 0 \\ 0 & 3 \end{bmatrix}, K = \begin{bmatrix} 12.1 & 7.1 & 0 \\ 3 & 1 & 2 \end{bmatrix}.$$

To synthesize the control law and the adaptation algorithm, we design the reference observer (14) with matrices

$$G_g = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -7 & 0 \\ 0 & 0 & -9 \end{bmatrix}, L_g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the disturbance observer (20) with matrices

$$G_\omega = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -3 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -11 & -6 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 0 \\ 0.1007 & 0.3022 & 0.4317 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1.7698 & 0.6906 & 0.1295 \end{bmatrix}.$$

By choosing $H(s) = \frac{1}{s+1}$, and performing the simulation, we obtain the results shown in Fig. 1 and Fig. 2.

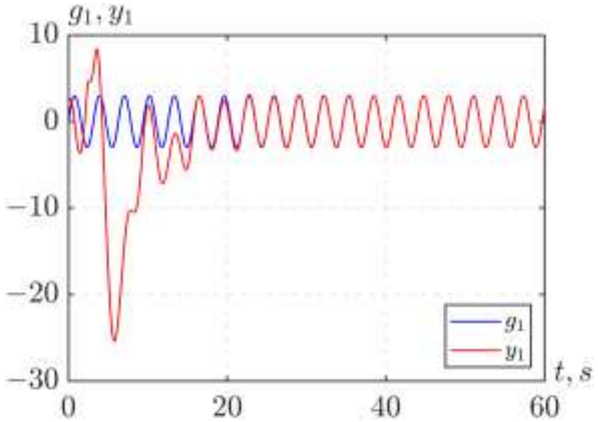

Fig. 1 - Transients in the closed-loop control system for channel 1 with $\mu_1 = 20$.

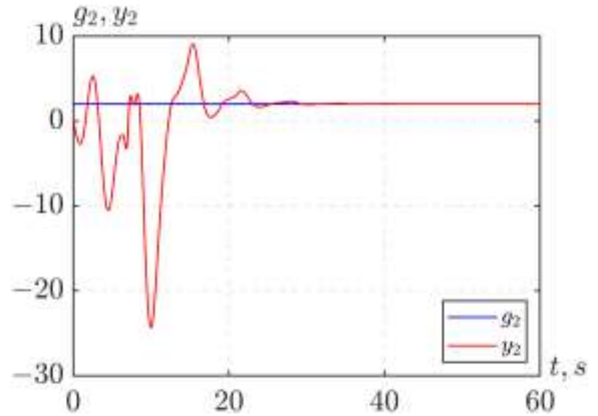

Fig. 2 - Transients in the closed-loop control system for channel 2 with $\mu_2 = 30$.

The analysis of the simulation results shows that the proposed method ensures that all signals in the closed-loop system remain bounded and satisfy requirement (2). The simulation results also indicate that the adaptation gains can be selected independently for each control channel.

IX. Conclusion

In this paper, a direct adaptive output-feedback control algorithm has been developed for a class of unstable multidimensional linear systems with distinct input delays, affected by multi-harmonic external disturbances whose parameters, including amplitude, frequency, phase, and bias, are unknown. The proposed control algorithm ensures convergence and stability of the output signals, while all closed-loop signals remain bounded. Moreover, the proposed method allows the adaptive algorithm to be designed independently for each control channel, thereby enabling the designer to select the adaptive gains independently in each channel according to the desired performance requirements.

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