Introduction to signal processing: sine wave and complex signals

E. Tikhonov and M. Sneps-Sneppe

***Abstract* — The article describes one of the powerful tools for the signals analysis while transmission – complex mathematics. Here the basic concepts (complex numbers, complex signal as expansion of a real signal to a complex-valued function) and some operations (exponentiation, absolute value and angular phase representation) are introduced.**

As an example complex mathematical methods are used for the case of classical electrical circuits with resistors, capacitors, coils. Actions can be performed both analytically and on a computer, using, for example, MatLab or any other computing software.

Illustrations of the signal passage through electrical circuits are supplied with a commented Matlab code, which is used to their producing.

The article is supposed to be used as a teaching aid, for self-learning the basics of the signal processing. It is considered as a first introduction part of a series on modern signal processing technologies.

Three main approaches to the exploring of the case are also demonstrated: ideological (experiment and its principal explanation), mathematical and computer modeling.

***Keywords* — complex numbers, complex signals, AC circuits, impedance, MatLab.**

I. INTRODUCTION

This article is an introduction to a series related to the signal processing tasks, actual for radio astronomy measurements and satellite data collection. The works are focused on applications of the “Ventspils International Radio Astronomy Center” of the Ventspils University of Applied Sciences, for example, for signal processing at the Reconfigurable Communication Subsystem, obtained from a 16-meter diameter parabolic antenna [1]. Digital section consists: in-flight reconfigurable FPGA baseband processor; subsystem managing microcontroller; radiation resistant FRAM memory. Analog section: 802.11a WLAN ADC/DAC frontend; 802.11a WLAN RF transceiver; RF PA and LNA.

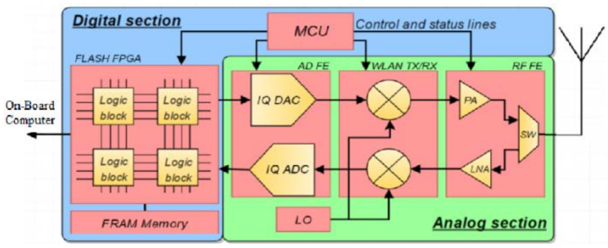

Fig. 1. Reconfigurable Communication Subsystem block diagram [1]

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The rest of the paper is the following. In Section 2, electrical response to alternative voltage of the basic elements like resistance, capacitor and coil is discussed. Section 3 suggests mathematical description of alternative current and Section 4 apply it to a previously explored elements behavior. Sections 5 and 6 introduce complex numbers and signals (like complex sine wave) correspondently. Section 6 applies this mathematical tool for each element. And in Section 7 this method is used to analyze RC and RL-circuits analytically and using MatLab computing.

II. RESISTANCE, CAPACITOR AND COIL (R, C, L) ALTERNATING CURRENT

We borrow the description of schemes from the textbook [2].

If we apply the voltage of a simple alternative (harmonic) current source to the basic electrical elements and then explore the current flowing through them using oscilloscope, we get different results, shown on Figure 2.

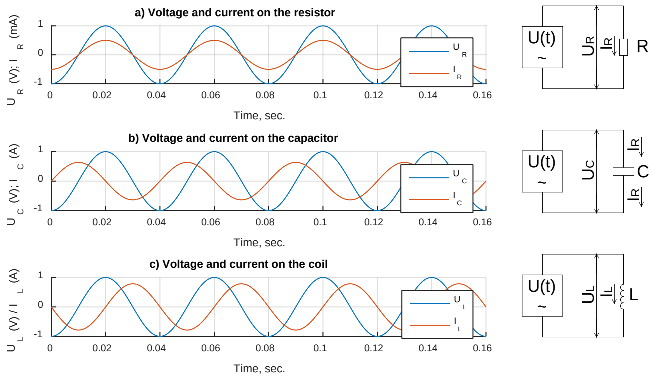

Fig 2. Time diagrams of alternating current on basic elements for applied harmonic voltage with amplitude 1 V, frequency 50 Hz simulation: resistor with resistance 2 kOm (a), capacitor with capacitance 5 mF (b), coil with inductance 10 mH (c) comparing with the voltage.

Matlab code:

```
% Time parameters
t0 = 0;             % start time
t1 = 8/50;          % finish time
steps = 1000;       % time accuracy step

% Cosine generator signal parameters
U0 = 1;             % amplitude
w0 = 50*pi;         % angular frequency
p0 = pi;            % initial phase

% RCL parameters
R = 2e3;            % 2 kOm
C = 5e-3;           % 5 mF
L = 10e-3;          % 10 mH

% Processing
```

```
t = linspace(t0, t1, steps);     % timeline

% voltage
U = U0*cos(w0*t + p0);

% resistance current
I_R = (U0/R)*cos(w0*t + p0);

% capacitor current
I_C = (U0/w0/C)*cos(w0*t + p0 + pi/2);

% coil current
I_L = (U0*L*w0)*cos(w0*t + p0 - pi/2);

% Plot a: resistor
figure();subplot(3,1,1);hold on; grid on;
plot(t, U);           % plot voltage
plot(t, 1e3*I_R);     % plot current
title('a) Voltage and current on the resistor');
xlabel('Time, sec.');ylabel('U_R (V); I_R (mA)');
legend('U_R', 'I_R');

% Plot b: capacitor
subplot(3,1,2);hold on; grid on;
plot(t, U);           % plot voltage
plot(t, I_C);         % plot current
title('b) Voltage and current on the capacitor');
xlabel('Time, sec.');
ylabel('U_C (V); I_C (A)');
legend('U_C', 'I_C');

% Plot c: coil
subplot(3,1,3);hold on; grid on;
plot(t, U);           % plot voltage
plot(t, I_L);         % plot current
title('c) Voltage and current on the coil');
xlabel('Time, sec.');ylabel('U_L (V) / I_L (A)');
legend('U_L', 'I_L');
```

In all the cases the shape of the current is the same as for initial voltage. But it is scaled in all of them and shifted in time for the capacitor and the coil.

If we do some more tests then we explore:

- time shift doesn't depend on the parameters of the element – just on the type;
- a increase in frequency leads to a proportional change in the scale of the current for the same coil (increases) and inductive (reduces) and doesn't affect the resistor;
- increasing the nominal of the element (resistance, capacitance, inductance) proportionally changes the scale: reduces (resistance, capacitance) or increases (coil).

Let's try to explain these properties and offer the way to predict the currents and voltages of the complex schemes with these types of elements.

III. Sine wave (alternative voltage)

The voltage of a harmonic current source (that we could interpret as a signal) has a shape of one of the basic analogous (i.e. continuous, without breaks) signals called "**sine wave**". It's a mathematical curve that describes a smooth periodic oscillation (most commonly used in communications). One of its common form is:

$$s(t) = A\cos(\omega t + \varphi_0) = A\cos\varphi(t)$$

where:

- "A" is called amplitude (corresponds to maximum value of s)
- "ω" is called angular frequency (corresponds to changes measured in radians per second". To use common frequency in Hz we should divide ω to 2π: $f = \frac{\omega}{2\pi}$
- "φ_0" is called initial phase (specified initial value of s for t=0, in radians). For example, if $\varphi_0 = \pi/2$ then $s(t) = A\cos(\omega t + {}^{\pi}/_{2}) = A\sin(\omega t)$
- "$\varphi = \varphi(t)$" is called full phase or current phase.

If all but *t* parameters always stay the same, we say that signal is stationary. In practice some of these parameters changes to add some more information – that is called "**modulation**".

As we have just one changing parameter (t or $\varphi(t)$ derived from it) we could make differens 2D visualizations of it:

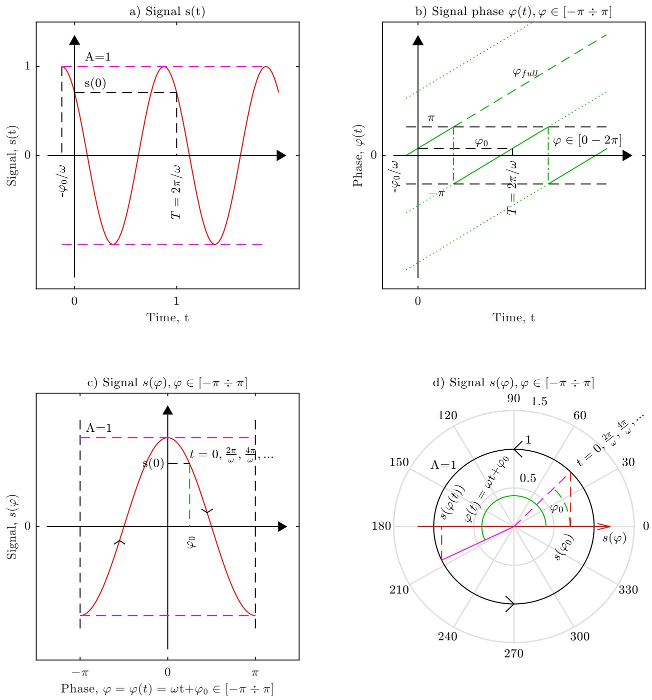

Fig. 3. Sine wave different visualizations: $s(t)$ as a time-function (a), phase $\varphi(t)$ as a time-function (b), $s(\varphi)$ as a phase-function (c) on Cartesian planes, and $s(\varphi)$ as a phase-function on the polar plane (d).

Because the sine/cosine takes on the same values for any $x = x_0 + 2\pi \times i$ (where i is integer), polar description (signal as a function of a phase) is not definitely reversible. For each function value there are multiple possible phases giving it. Several interpretations of that are possible:

- suggest that phase abruptly decreases to 0 when reaches 2π (see b, c on Figure 3);
- imagine sine function as an infinite number of phases (indistinguishable), that are "active" when their value is in range [0 - 2π] (see b on Figure 3);
- consider only representation in polar coordinates (see d on Figure 3) – with obvious round cycle;

The interesting and important fact is that rate of sine wave change is also sine wave (scaled and time shifted):

$$\frac{ds(t)}{dt} = \frac{d\cos(\omega t + \varphi_0)}{dt} = -A\omega\sin(\omega t + \varphi_0) = A\omega\cos\left(\frac{\pi}{2} + \omega t + \varphi_0\right) = \omega s\left(t + \frac{\pi}{2\omega}\right)$$

IV. Sine wave passing throught resistor, capacitor, coil in circuits

When sine wave signal (sinusoidal alternating voltage) is applied to active (ohmic) resistance, value current is proportional to fall of potential on this element and has the zero time shift, i.e., they coincide in phase (see Figure 2a and 4a).

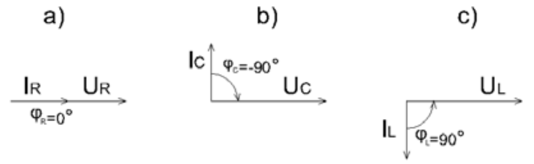

Fig. 4. Vector diagram for of alternating voltage and current value on basic elements: resistor (a), capacitor (b), coil (c)

This is displayed in physics as Ohm’s law as a definition for resistor resistance:

$$U_R(t) = R \times I(t)$$

The situation is different when an external sine voltage is applied to a capacitor (two metal plates separated by a dielectric). Through an ideal capacitor the flow of electrons (that is an electric current in fact) does not pass. But when external voltage is changed, the first capacitor’s plane immidiately accumulates additional electrons and the second plate emits them compensating the new voltage level. The greater the change in voltage (over time), the greater is the number of accumulated and emmited electons (over that time). In oher words, the current is proportional to a voltage change rate.

Therefore the “charging” current flows in the circuit before and after the capacitor during the voltage growth to the peak. When external voltage begins to decrease (from the maximum value to 0), a similar reversed “discharge” apears. Then the same process occurs in the opposite direction (to a negative maximum and again back to zero).

Mathematically, this current dependence on changes in external voltage is written as the definition of inductance (assuming the current is linearly dependent on voltage change rate):

$$I_C(t) = C\frac{dU_C(t)}{dt} = C\frac{d\cos(\omega t + \varphi_0)}{dt} = \omega C \times U(t + \frac{\pi}{2\omega})$$

Or:

$$U(t) = \frac{1}{\omega C} I_C\left(t - \frac{\pi}{2\omega}\right)$$

Thus, the sinusoidal form of the signal is preserved. This fact has fundamental importance for the linearity of systems and for using of complex voltages that can be represented as sum of sinusoids due to the Fourier transform. But charging/discharging of a capacitor causes a shift between the voltage and the current (phase shift $\varphi = 90°$ corresponds to $-\frac{\pi}{2\omega}$ time shift), which is shown in the timeline (Figure 2b) and vector diagrams (Figure 4b). In addition, the amplitude of current and voltage does not depend only on the capacitance C, but also on the frequency of change of the external voltage.

Similarly, when a flow of electrons (electical current) starting to move throught the coil under the force of the applied external voltage $U(t)$, it generates a magnetic field around it according to Maxwell's laws. This magnetic field creates a self-induction vsoltage $E(t)$ in the coil, directed against the external voltage and equals to that to that slowing down the current rise on the coil. When the external field becomes zero, the current arising stops (and so the self-induction does), the current has the maximum value. Then the current decreasing to zero begins. After that the same cycle in the negative values takes the place.

In mathematical form:

$$E(t) = -U(t) = -L\frac{dI(t)}{dt} = -\cos(\omega t + \varphi)$$

So,

$$I(t) = -\frac{1}{L\omega}\sin(\omega t + \varphi) = \frac{1}{L\omega}\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = \frac{1}{L\omega}U\left(t - \frac{\pi}{2\omega}\right)$$

So the current is ahead of the voltage in the coil in 90 degrees (or $\frac{\pi}{2\omega}$ seconds), or we could say the voltage is behind of the current. Sinusoidal waveform is preserved, the proportionality coefficient of the amplitudes of voltage and current also depends not only on the characteristics of the coil (inductance), but also on the frequency of the signal, that is, the rate of its change:

$$U_L(t) = L\omega \times I_L\left(t + \frac{\pi}{2\omega}\right)$$

Composite schemes with these elements can be analyzed using vector form (analytically or graphically), or the complex representation of numbers and signals could be used to keep the convenient simple form of Ohm's law:

$$U_C(t) = X_C \times I(t)$$
$$U_L(t) = X_L \times I(t)$$

V. Complex numbers

Complex number is the mathematical abstraction, that has the form: $x = a + jb$, where a and b are real numbers (called “real part” and “imaginary part” respectively) and j is a solution of the equation $j^2 = 1$. No real number correspond a j, so it is called “imaginary”. Imaginary and real part of complex number are completely independent (one cannot be received from the other with any mathematical operations).

Complex numbers are the powerful tool to operate with different physical processes. Basic operations with these type of numbers are obvious from the definition (addition, multiplication and so on).

Real part of x (a) usually noted $Re(x)$ or $Re\,x\;x$ and imaginary part (b) noted $Im(x)$or $Im\,x$. The number $x^* = a - jb = Re(x) - jIm(x)$ is called “complex conjugate”. Because of x is a function of 2 independent parameters (a and b), it could be shown on the “complex” plane:

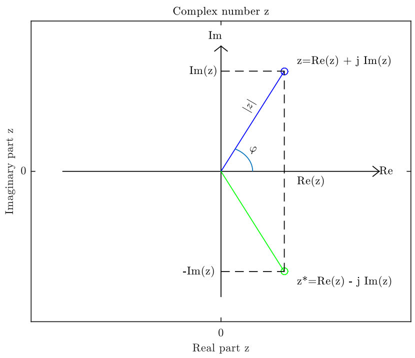

Fig. 5. Complex number $z = Re(z) + jIm(z)$ on complex plane (polar properties: length $|z| = \sqrt{Re(z)^2 + Im(z)^2}$, phase $\varphi = arctg(^{Im\,z}/_{Re\,z})$) and complex conjugate $z^* = Re(z) - jIm(z)$.

Also it could be written as a vector:

$$\binom{a}{b} = \binom{Re(x)}{Im(x)}$$

for processing as common Cartesian vectors. Alternative we could use the polar coordinates – length and angle:

$$\binom{|x|}{\varphi} = \binom{\sqrt{a^2 + b^2}}{arctg(^b/_a)}$$

In this representation we could use all the methods of analytical geometry.

Exponentiation: Euler's formula states that, for any real number x:

$$e^{jx} = \cos x + j \sin x$$

where e is the base of the natural logarithm.

Complex conjugate for this number is $\cos x - j \sin x = \cos(-x) + j \sin(-x) = e^{-jx}$.

This type of representation of trigonometric functions makes many operations with them easier.

VI. Complex Sine Wave

Sine wave could be written as

$$s(t) = A\cos(\omega t + \varphi_0) = Re\left(A \times e^{j(\omega t + \varphi_0)}\right)$$
$$= Re(A \times e^{j\varphi_0} \times e^{j\omega t}) = Re(Z \times e^{j\omega t})$$
$$= Re\big(Z(t)\big),$$

or:

$$s(t) = \frac{Re\,Z(t) + Im\,Z(t)}{2},$$

where $Z = A \times e^{j\varphi_0}$ called “complex amplitude” and $Z(t) = Z \times e^{j\omega t} = A \times e^{j\varphi_0} \times e^{j\omega t} = A(\cos(\omega t + \varphi_0) + j\sin(\omega t + \varphi_0))$ called “complex sine wave”.

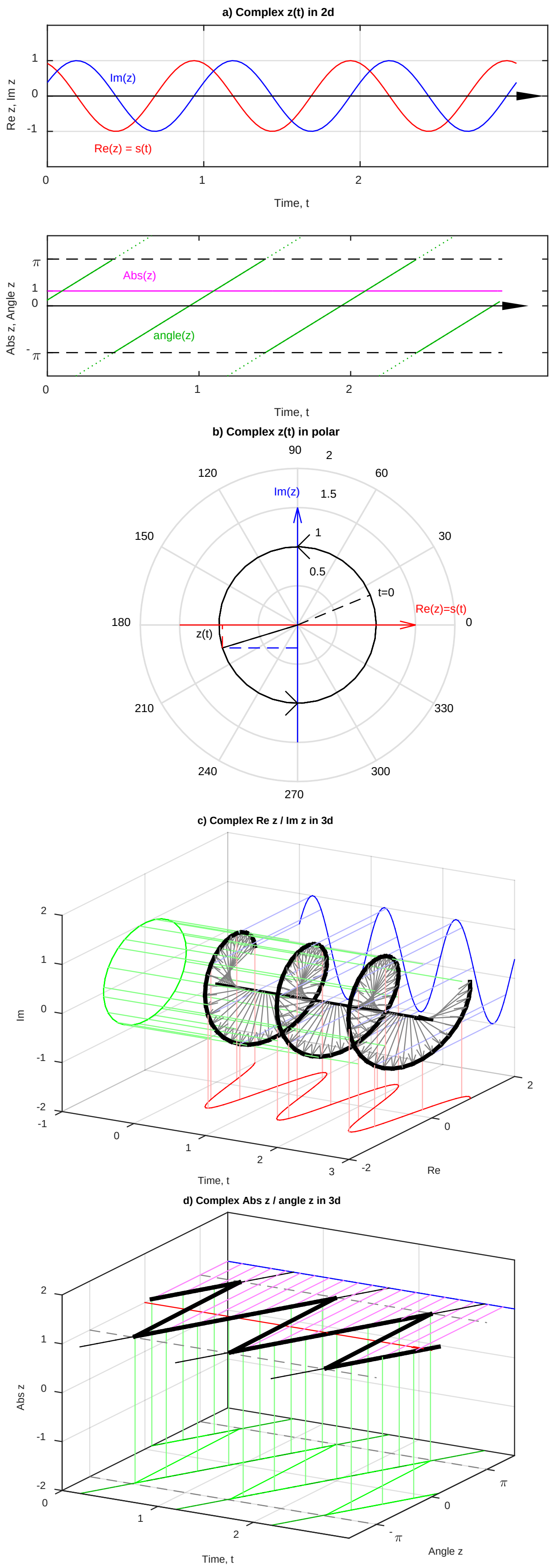

Fig. 6. Complex sine wave $z(t) = Ae^{j\varphi_0} \times e^{j\omega t}$ (with $A = 1$, $\omega = 2\pi$, $\varphi_0 = ^{\pi}/_8$): Cartesian (a), polar (b), three-dimensions Re-Im (c) and Abs-Angle (d) representations. Real part corresponds to real sine wave $s(t) = cos(\omega t + \varphi_0)$ with the same parameters (colored red)

Matlab code (some design features are skipped):

```
% Complex sine wave
figure('units', 'normalized', 'outerposition', [0
0 0.8 1]);

%% Constants definition
A = 1;                  % amplitude
w = 2*pi;               % angle velocity
ph0 = pi/8;             % initial phase

scale = 2;
N_counts = 100;         % accuracy
t_max = 3;              % max time to plot
t_min = 0;              % min time to plot

%% Processing
t = linspace(0, t_max, N_counts);  % timeline
t_step = (t_max-t_min) / N_counts; % time step

sine_c = A*exp(j*(w*t+ph0)); % complex sine wave
Re = real(sine_c);           % real part of z(t)
Im = imag(sine_c);           % imaginary of z(t)

o = @(t) zeros(size(t));     % zero-vector
e = @(t) ones(size(t));      % one-vector
ph_full = @(t) w*t+ph0;      % full-phase

%% Figure a1: Re/Im in 2d
subplot(4, 2, 1); hold on; grid on;

plot(t, Re, 'r');            % plot Re
plot(t, Im, 'b');            % plot Im

%% Figure a2: Abs / angle in 2d
subplot(4, 2, 3); hold on;
plot(t, abs(sine_c), 'm');   % plot Abs

% plot angle
for i = ceil((w*t_min+ph0)/2/pi)-
1:floor((w*t_max+ph0)/2/pi)
    % before and after pi
    t_start  =  min(max((pi*(2*i-1)-ph0)/w,  t_min),
t_max);
    t_end   =   max(min((pi*(2*i+1)-ph0)/w,   t_max),
t_min);
    t_temp = t_min:t_step:t_max;
    plot(t_temp,ph_full(t_temp)-2*pi*i,    'Color',
[0 0.7 0], 'LineStyle', ':');
    % -pi - pi
    t_temp = t_start:t_step:t_end;
    plot(t_temp,ph_full(t_temp)-2*pi*i,    'Color',
[0 0.7 0], 'LineStyle', '-');
end

%% Figure b: Abs/angle in 2d
subplot(4, 2, [5,7]);
scale_trick=polar([0 0], [0 A*2]);
% polar scale trick
set(scale_trick, 'Visible', 'off'); hold on;

polar(angle(sine_c), abs(sine_c), 'k');    % Polar

%% Figure c: Re/Im in 3d
subplot(4, 2, [2,4]); hold on; grid on;
view(30, 26);

% Complex sine curve
plot3(t, Re, Im, 'k', 'LineWidth', 3);
plot3(t, e(t)*scale, Im, 'b');  % Im projection
plot3(t, Re, -e(t)*scale, 'r'); % Re projection
plot3(-1*e(t), Re, Im, 'g');    % Polar projection
% polar vectors
quiver3(t, o(t), o(t), o(t), Re, Im, 0, 'Color',
[0.5 0.5 0.5]);

%% Figure d: abs / angle
subplot(4, 2, [6,8]); hold on; grid on; view(30,
26);

% Compex sine wave
plot3(t,    angle(sine_c),    abs(sine_c),    'k',
'LineWidth', 3);
% Abs projection
plot3(t, e(t)*1.5*pi, abs(sine_c), 'b');
% Angle projection
plot3(t, angle(sine_c), -e(t)*scale, 'g');
```

Matlab code for the Figutre 7 (some design features are skipped):

```
% Complex sine forming
figure('units', 'normalized', 'outerposition', [0
0 0.5 1])

%% Constants definition
w = 2*pi;            % angle velociry
ph0 = pi/8;          % initial phase
A = 1;               % amplitude
t_max = 2;           % time to plot
N_counts = 60;       % accuracy

%% Signal
t = linspace(0, t_max, N_counts);% timeline
o = zeros(size(t));              % zero-vector

sine_c = A*exp(j*(w*t+ph0));    % comlex sine wave
sine_r = real(sine_c);
sine_i = imag(sine_c);

%% Figure a: real
subplot(2, 2, 1); hold on; grid on;
xlim([0 t_max]); ylim([-1 1]);
zlim([-1 1]); view(-40,40);

plot3(t, sine_r, o, 'r');       % sine curve
quiver3(t, o, o, o, sine_r, o, 0, 'r'); % ar-field

%% Figure b: imaginary part
subplot(2, 2, 2); hold on; grid on;
xlim([0 t_max]); ylim([-1 1]);
zlim([-1 1]); view(-40,40);

plot3(t, o, sine_i, 'b');      % sine curve
quiver3(t, o, o, o, o, sine_i, 0, 'b'); % ar-field

%% Figure c: real + imaginary
subplot(2, 2, 3); hold on; grid on;
xlim([0 t_max]); ylim([-1 1]);
zlim([-1 1]); view(-3,20);

plot3(t, sine_r, o, 'r');     % sine curves
plot3(t, sine_r, sine_i, 'b');

quiver3(t, o, o, o, sine_r, o, 0, 'r'); % ar-field
quiver3(t, sine_r, o, o, o, sine_i, 0, 'b')

%% Figure d: result polar
subplot(2, 2, 4); hold on; grid on;
xlim([0 t_max]); ylim([-1 1]);
zlim([-1 1]); view(-30,15);

quiver3(t, o, o, o, sine_r, sine_i, 0, 'k');
plot3(t, sine_r, sine_i, 'k');    % arrow field
```

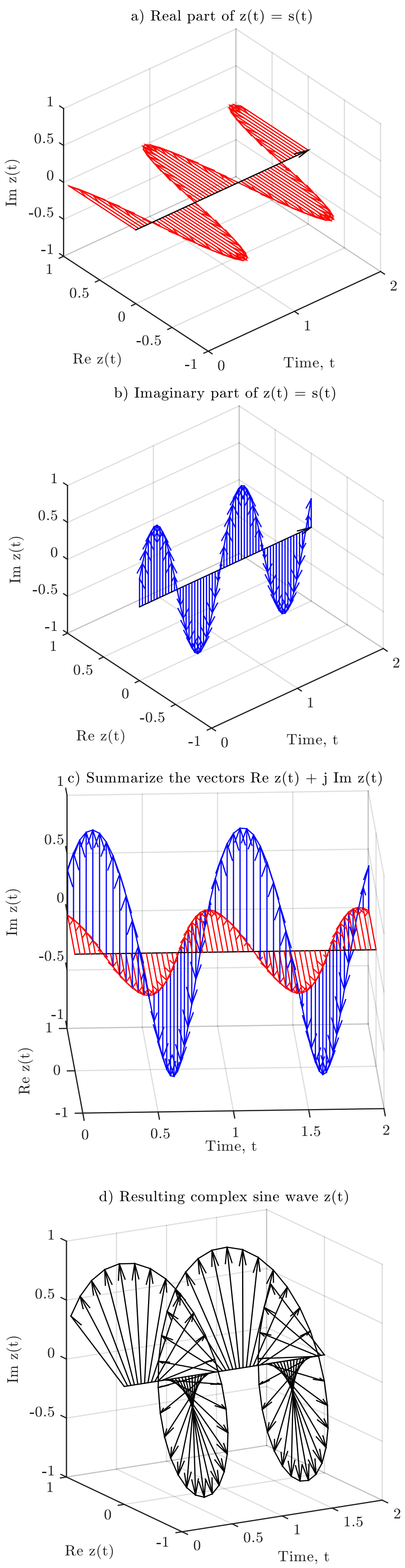

Fig. 7. Complex sine wave forming $z(t) = A\ Re\ z(t) + j\ A\ Im\ z(t) = A\ cos(\omega t + \varphi_0) + j\ A\ sin(\omega t + \varphi_0)$ forming process (with $\omega = 2\pi$, $\varphi_0 = {}^{\pi}/_{8}$)

So, the real and imaginary parts in complex sine wave are not independent. They are bind so that imaginary part exactly repeats the real part with a phase shift ${}^{\pi}/_{2}$ corresponding to time delay ${}^{\pi}/_{2\omega}$.

The real part of complex sine wave corresponds to a real sine signal. And imaginary part is synchronous addition to that, and is needed to provide this in a complex space.

VII. Complex impedance of the resistor, capacitor, coil

The phase difference between the voltage and the current of the coil or capacitor can be taken into account, while maintaining the convenient form of Ohm's law (voltage equal to the current multiplied by the resistance). Of couse, we should imagine complex sine wave generator $U(t)$ to provide that it's real part will correspond to a real sine signal.

We assume that the resistance has a phase:

$$R_C = X_C = |X_C|e^{j\varphi}$$

The phase delay of the voltage on the capacitor by -90 degrees from the current ($\varphi = -\pi/2$, that is, 1/4 of the full period) means:

$$U_C = I_C \times |X_C|e^{j\varphi} = I_C \times |X_C|e^{j\left(-\frac{\pi}{2}\right)} = I_C \times \frac{1}{\omega C}e^{-j\frac{\pi}{2}}$$

According to Euler's formula:

$$e^{j\left(-\frac{\pi}{2}\right)} = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = -j = \frac{1}{j}$$

Therefore, Ohm's law for a capacitor can be written in the form of capacitance:

$$U_C(t) = X_C \times I_C(t) = \frac{1}{j\omega C} \times I_C(t) = -j\frac{1}{\omega C} \times I_C(t)$$

Similarly, due to the fact that the amplitudes of the voltage and current on the coil are related as $|X_L| = \omega L$, and the current shift from the voltage is +90 degrees:

$$U_L(t) = X_L \times I_L(t) = j\omega L \times I_L(t)$$

These complex resistance analogs are called “reactance” - resistance that a capacitor or inductance exerts on alternating current. Reactive resistance in contrast to active resistance does not consume energy, but accumulates it and returns it to the circuit. Any real elements contain both active and reactive (inductive and capacitive) resistance and can be represented as their sum, the so-called impedance[3]:

$$Z = (Re\ Z + j\ Im\ Z) = R + jX$$

$$U_A(t) = Z \times I_A(t) = (Re\ Z + j\ Im\ Z) \times I_A(t) = |Z|e^{\varphi_Z} \times I_A(t)$$

The impedance at $X = 0$ corresponds to the active resistance R, at $X > 0$ has a capacitive and when $X<0$ – inductive type.

Calculations of parallel and serial connections of active elements (coils and capacitors), as well as passive (resistors), can be used in this complex form, as well as in the vector form:

$$Z_{series} = \sum_{i=1}^{n} Z_i$$

$$\frac{1}{Z_{parallel}} = \sum_{i=1}^{n} \frac{1}{Z_i}$$

Matlab code: complex method gives the same result as on Figure 2.

```
% Time parameters
t0 = 0;              % start time
t1 = 8/50;           % finish time
steps = 1000;        % time accuracy step

% Cosine generator signal parameters
U0 = 1;              % amplitude
w0 = 50*pi;          % angular frequency
p0 = pi;             % initial phase

% complex sine wave
c = @(t) U0*exp(j*(w0*t + p0));

% RCL parameters
R = 2e3;             % 2 kOm
C = 5e-3;            % 5 mF
L = 10e-3;           % 10 mH

% Impedance
X_R = R;
X_C = 1/(j*w0*C);
X_L = j*w0*L;

% Processing
t = linspace(t0, t1, steps);

U = c(t);            % voltage
I_R = U/X_R;         % resistance current
I_C = U/X_C;         % capacitor current
I_L = U/X_L;         % coil current

% Plot a: resistor
figure();subplot(3,1,1);hold on; grid on;
plot(t, real(U));          % plot voltage
plot(t, 1e3*real(I_R));    % plot current
title('a) Voltage and current on the resistor');
xlabel('Time, sec.');
ylabel('U_R (V); I_R (mA)');
legend('U_R', 'I_R');

% Plot b: capacitor
subplot(3,1,2);hold on; grid on;
plot(t, real(U));          % plot voltage
plot(t, real(I_C));        % plot current
title('b) Voltage and current on the capacitor');
xlabel('Time, sec.');
ylabel('U_C (V); I_C (A)');
legend('U_C', 'I_C');

% Plot c: coil
subplot(3,1,3);hold on; grid on;
plot(t, real(U));          % plot voltage
plot(t, real(I_L));        % plot current
title('c) Voltage and current on the coil');
xlabel('Time, sec.');
ylabel('U_L (V) / I_L (A)');
legend('U_L', 'I_L');
```

VIII. Example. Analysis of RC and RL circuits

A. RC circuit

Figure 7 shows the series connection of resistance R and capacitor C. The task is to determine the particular voltages U_R and U_C. The circuit has a resistance R and a reactance capacitor X_C, which introduces a phase shift between current I and voltage U.

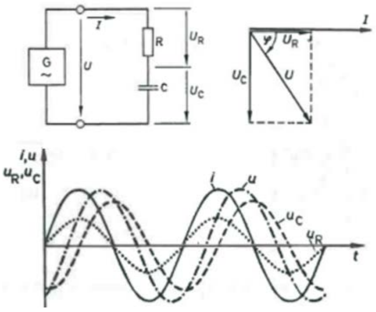

Fig. 8. RC circuit and its time and vector diagrams

The time diagram shows the four curves I, U, U_C and U_R. It is seen that the current I is ahead of the voltage U_C by 90^0, but the phase shift between the common voltage U and the current I is in the range between 0 and 90 degrees.

Summary impedance:

$$Z = Z_R + Z_C = R - j\frac{1}{\omega C}$$

Though voltage in the circuit is U, the current is

$$I_R = I_C = \frac{U}{Z} = \frac{U}{R - j\frac{1}{\omega C}}$$

Voltage drop on resistor is:

$$U_R = R \times I_R = U\frac{R}{R - j\frac{1}{\omega C}} = U\frac{1}{1 - j\frac{1}{\omega RC}} = U\frac{\omega RC}{\omega RC - j}$$
$$= U\left(1 + \frac{1}{(\omega RC)^2 - 1} + j\frac{\omega RC}{(\omega RC)^2 - 1}\right)$$

Voltage drop on capacity is:

$$U_C = U - U_R = U\left(\frac{1}{1 - (\omega RC)^2} - j\frac{\omega RC}{(\omega RC)^2 - 1}\right)$$

So the real voltages are:

$$Re\ U_R = U\left(1 + \frac{1}{(\omega RC)^2 - 1}\right)$$
$$Re\ U_C = U\frac{1}{1 - (\omega RC)^2}$$

B. RL-circuit

Figure 8 shows the series connection of resistance R and inductance L. The task is to determine the particular voltages U_R and U_L. The circuit has a resistance R and a reactance inductance X_L, which also introduces a phase shift between current I and voltage U, but it has a different character.

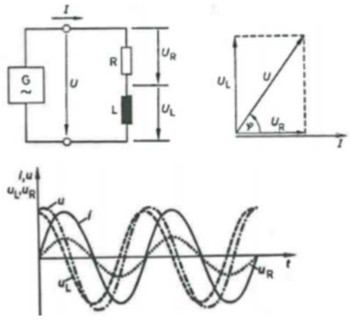

Figure 9. RL-circuit and its time and vector diagrams

Figure 8 shows that the reactive voltage U_L is 90^O ahead of the current I, the active voltage U_R is in the same phase as the current I, the phase shift between the common voltage U and the current I is in the range between 0 and 90 degrees.

Summary impedance:

$$Z = Z_R + Z_L = R + j\omega L$$

Though voltage in the circuit is U, the current is

$$I_R = I_L = \frac{U}{Z} = \frac{U}{R + j\omega L}$$

Voltage drop on resistor is:

$$U_R = R \times I_R = U\frac{R}{R + j\omega L} = U\frac{1}{1 + j\frac{\omega L}{R}}$$

$$= U\left(\frac{1}{1 - \left({}^{\omega L}/_{R}\right)^2} - j\frac{{}^{\omega L}/_{R}}{1 - \left({}^{\omega L}/_{R}\right)^2}\right)$$

Voltage drop on inductance is:

$$U_L = U\frac{j\omega L}{R + j\omega L} = U\left(\frac{R\left({}^{\omega L}/_{R}\right)^2}{1 - \left({}^{\omega L}/_{R}\right)^2} + j\frac{\omega L}{1 - \left({}^{\omega L}/_{R}\right)^2}\right)$$

So the real voltages are:

$$Re(U_R) = \frac{U}{1 - \left({}^{\omega L}/_{R}\right)^2}$$

$$Re(U_L) = U\frac{R\left({}^{\omega L}/_{R}\right)^2}{1 - \left({}^{\omega L}/_{R}\right)^2}$$

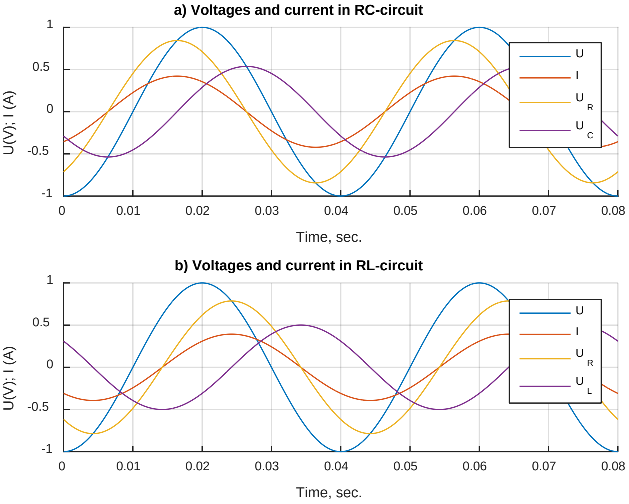

Figure 10. Time diagrams simulation for RC (a) and RL (b) circuits for applied harmonic voltage with amplitude 1 V, frequency 50 Hz: resistor with resistance 2 Om, capacitor with capacitance 5 mF, coil with inductance 10 mH

Matlab code:

```
% Time parameters
t0 = 0;              % start time
t1 = 4/50;           % finish time
steps = 1000;        % time accuracy step

% Cosine generator signal parameters
U0 = 1;              % amplitude
w0 = 50*pi;          % angular frequency
p0 = pi;             % initial phase

% complex sine wave
c = @(t) U0*exp(j*(w0*t + p0));

% RCL parameters
R = 2;               % 2 Om
C = 5e-3;            % 5 mF
L = 10e-3;           % 10 mH

% Impedance
X_R = R;
X_C = 1/(j*w0*C);
X_L = j*w0*L;

% Processing
t = linspace(t0, t1, steps);

U = c(t);                 % voltage

I_RC = U/(X_R+X_C);       % RC-current
U_RC_R = X_R * I_RC;      % U_R in RC circuit
U_RC_C = X_C * I_RC;      % U_C in RC circuit

I_RL = U/(X_R+X_L);       % RC-current
U_RL_R = X_R * I_RL;      % U_R in RL circuit
U_RL_L = X_C * I_RL;      % U_C in RL circuit

% Plot a: RC
figure();subplot(2,1,1); hold on; grid on;
plot(t, real(U));         % plot voltage
plot(t, real(I_RC));      % plot current
plot(t, real(U_RC_R));    % plot U_R
plot(t, real(U_RC_C));    % plot U_C
title('a) Voltages and current in RC-circuit');
xlabel('Time, sec.');ylabel('U(V); I (A)');
legend('U', 'I', 'U_R', 'U_C');

% Plot b: RL
subplot(2,1,2); hold on; grid on;
plot(t, real(U));         % plot voltage
plot(t, real(I_RL));      % plot current
plot(t, real(U_RL_R));    % plot U_R
plot(t, real(U_RL_L));    % plot U_L
title('b) Voltages and current in RL-circuit');
xlabel('Time, sec.');ylabel('U(V); I (A)');
legend('U', 'I', 'U_R', 'U_L');
```

IX. Conclusion

The basic concept tool of signal processing analysis was discussed: the complex mathematical method basics, like complex sine wave signal and related operations. The in following parts some technologies based on this idea are planned to discuss: spectrum, analytical (complex) representation of a signal, information (digital) signal, modulations, spectrum modifications (like Direct Sequence Spread Spectrum, Frequency Hopping, Time Hopping, Chirp Spread), etc.

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